Gravitational Waves and the Sagnac Effect

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Abstract

We consider light waves propagating clockwise and other light waves propagating counterclockwise around a closed path in a plane (theoretically with the help of stationary mirrors). The time difference between the two light propagating path orientations constitutes the Sagnac effect. The general relativistic expression for the Sagnac effect is discussed. It is shown that a gravitational wave incident to the light beams at an arbitrary angle will *not* induce a Sagnac effect so long as the wave length of the weak gravitational wave is long on the length scale of the closed light beam paths. The gravitational wave induced Sagnac effect is thereby null.

1 Introduction

With the help of reflecting mirrors, one may propagate a light signal in a clockwise closed path (C). Similarly, a light signal can be propagated in the counterclockwise closed path ($-C$). The time of propagation in the clockwise orientation need not be equal to the time of propagation in the counterclockwise direction. The time difference $\Delta t = t[C] - t[-C]$ between the two possible oriented paths constitutes the Sagnac effect. The original experimental verification[1] of this effect required that the mirrors be fixed to a rotating table. Two light beams going in opposite directions exhibited a phase interference depending linearly on the angular velocity of the table. If the path C is the boundary of a plane open surface Σ, i.e. $C = \partial\Sigma$, then the time difference is given by

$$\Delta t = \frac{4}{c^2}\int_{\Sigma} \mathbf{\Omega} \cdot d\mathbf{\Sigma}, \tag{1}$$

where $\mathbf{\Omega}$ is the axial vector angular velocity. The Sagnac effect forms the basis of practical engineering devices measuring rotational velocities.

More recently, proposals that the Sagnac effect may be made the basis of gravitational wave detection[2, 3, 4, 5, 6] have been of considerable recent interest[7, 8, 9, 10]. The experimental technique envisions a *Laser Interferometer Space Antenna* (LISA). The general idea[12, 13, 14] is illustrated in *schematic* form in Fig.1. Three enclosed reflecting mirrors float in space forming a triangle.

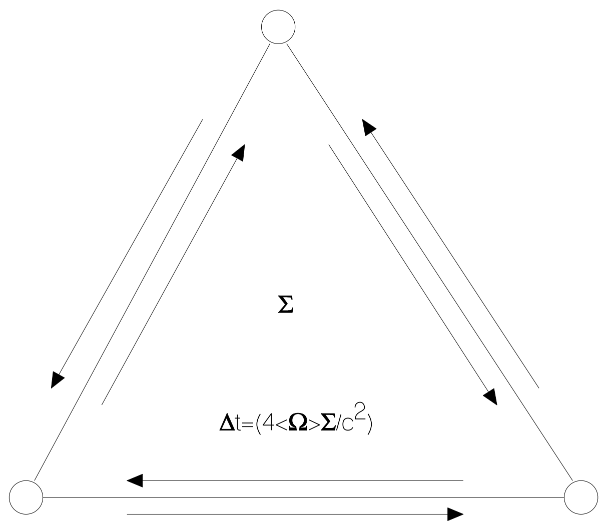

Figure 1: In a triangular geometrical arrangement contemplated for LISA, one may consider the mean flux of the local axial angular velocity vector $\mathbf{\Omega}$ through the area Σ. In detail, $\int_\Sigma \mathbf{\Omega} \cdot d\mathbf{\Sigma} = \langle \Omega \rangle \, \Sigma$ yields the time difference between the clockwise and counterclockwise light ray paths according to the Sagnac rule $\Delta t = (4 \langle \Omega \rangle \, \Sigma / c^2)$. A plane polarized gravitational wave does indeed produce a local rotational axial vector $\mathbf{\Omega}$. However, in the limit in which the wavelength λ of the gravitational wave obeys $\lambda^2 >> \Sigma$ and to lowest order in the gravitational wave curvature, one finds that $\langle \Omega \rangle \approx 0$. The Sagnac effect is thereby *null*.

Clockwise and counterclockwise beams of light propagate on a path C which bounds a triangular area Σ. Let L denote the length scale of the sides of the triangle and let λ denote the wavelength of a polarized gravitational wave incident upon the triangle from a direction described by a unit vector $\mathbf{n}$. It is proposed that the Sagnac time difference in Eq.(1) in the limit $\lambda >> L$ can be employed to probe the local rotational velocity $\mathbf{\Omega}$ produced by the gravitational wave. The principle is that one may measure the flux of the rotational velocity $\int_\Sigma \mathbf{\Omega} \cdot d\mathbf{\Sigma}$.

In order to see what is involved, let us consider an electromagnetic and gravitational wave analogy. Suppose a measuring device for magnetic flux $\Phi = \int_\Sigma \mathbf{B} \cdot d\mathbf{\Sigma}$ and further suppose that one wished to detect a plane polarized electromagnetic wave by measuring the magnetic field $\mathbf{B}$ via a magnetic flux reading device. If the electromagnetic wavelength and the flux area obey

$$\lambda^2 >> \Sigma, \tag{2}$$

then the magnetic flux $\Phi = \int_\Sigma \mathbf{B} \cdot d\mathbf{\Sigma}$ meter could serve as a polarized electromagnetic wave detector provided that the direction $\mathbf{n}$ and polarization $\mathbf{e}$ of the electromagnetic wave were in appropriate directions relative to the flux area Σ. Similarly, for a gravitational wave incident upon the LISA arrangement pictured in Fig.1, the Sagnac time difference $\Delta t = (4/c^2) \int_\Sigma \mathbf{\Omega} \cdot d\mathbf{\Sigma}$ measures the flux of a gravitational wave induced rotational velocity $\mathbf{\Omega}$ through the area Σ. However, it will be shown in what follows, that if the gravitational wavelength λ obeys Eq.(2), then to linear order in the gravitational wave curvature tensor the Sagnac $\Delta t \approx 0$. The LISA gravitational wave induced Sagnac effect is thereby theoretically null.

In Sec.2, the general expression for the Sagnac effect in a quasi-static metric is derived. A (possibly strong) gravitational plane wave solution to the vacuum Einstein equations is explored in Sec.3. An expression for the local rotational velocity $\mathbf{\Omega}$ due to a gravitational wave is derived. In Sec.4, the Sagnac delay time Δt due to a quasi-static gravitational wave is computed and shown to be null in the quasi-static limit of Eq.(2). In the concluding Sec.5, the symmetry principles involved in the theory are discussed from a physical viewpoint.

2 Sagnac Effect

Consider a general proper time metric in general relativity

$$-c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{3}$$

wherein $x = (x^1, x^2, x^3, x^0) = (r^1, r^2, r^3, ct) = (\mathbf{r}, ct)$ denotes space time coordinates. Employing the definitions

$$\begin{aligned}
g_{00}(x) &= -h(\mathbf{r}, t), \\
g_{i0}(x) &= g_{0i}(x) = \frac{h(\mathbf{r}, t) u_i(\mathbf{r}, t)}{c} , \\
g_{ij}(x) &= \gamma_{ij}(\mathbf{r}, t) - h(\mathbf{r}, t) \left(\frac{u_i(\mathbf{r}, t) u_j(\mathbf{r}, t)}{c^2} \right) ,
\end{aligned} \tag{4}$$

yields the metric expressions

$$
\begin{aligned}
c^2 d\tau^2 &= c^2 d\tilde{t}^{\,2} - d\ell^2, \\
cd\tilde{t} &= \sqrt{h}\left(cdt - \frac{u_i dr^i}{c}\right) = \sqrt{h}\big(cdt - (\mathbf{u}\cdot d\mathbf{r}/c)\big) \\
d\ell^2 &= \gamma_{ij} dr^i dr^j = \gamma : d\mathbf{r}d\mathbf{r}.
\end{aligned} \tag{5}
$$

The metric $\{\gamma_{ij}\}$ will be used in what follows to describe the possibly curved spatial geometry via the distance $d\ell$.

A light signal connecting two neighboring events is characterized by a zero proper time interval or equivalently by a propagation light speed c; i.e.

$$
d\tau = 0 \quad \text{implies} \quad \left|\frac{d\ell}{d\tilde{t}}\right| = c. \tag{6}
$$

The following is worthy of note: If one reverses the spatial displacement between two neighboring events $d\mathbf{r} \to -d\mathbf{r}$ both connected by light signals, then the change in the effective time of those signals obeys $\Delta(dt) = 2(\mathbf{u}\cdot d\mathbf{r}/c^2)$. In more detail, let us suppose that light (with the help of perfectly reflecting mirrors) moved around a closed path C. Let us also suppose that different light moved "backwards" around the same closed path. If the metric is *quasi-static* on the time scale of the round trip C and the reversed round trip $-C$, then the time difference between the two trips obeys

$$
\Delta t = \frac{1}{c^2}\left(\oint_C - \oint_{-C}\right) u_i dr^i = \frac{2}{c^2}\oint_C u_i dr^i \equiv \frac{2}{c^2}\oint_{C=\partial\Sigma} \mathbf{u}\cdot d\mathbf{r}, \tag{7}
$$

where we have noted that the closed curve $C = \partial\Sigma$ is the boundary of an open surface Σ. Using the spatial metric $d\ell^2 = \gamma_{ij} dr^i dr^j$ $(\gamma = \det\gamma_{ij})$, allows us to define $\mathbf{\Omega}$ via

$$
\partial_i u_j - \partial_j u_i = \sqrt{\gamma}\,\epsilon_{ijk}(\mathbf{curl\ u})^k, \qquad d\mathbf{\Sigma}_k = \frac{1}{2}\sqrt{\gamma}\,(dr^i \wedge dr^j)\epsilon_{ijk}, \tag{8}
$$

$$
\mathbf{\Omega} = \frac{1}{2}\mathbf{curl\ u}. \tag{9}
$$

Theorem 1: (The Sagnac Effect)

Consider a light beam moving clockwise and another light beam moving counterclockwise around a closed curve boundary $C = \partial\Sigma$ of an open surface Σ in a quasi-static gravitational metric. The difference in the circulation times is proportional to the flux of the local angular velocity $\mathbf{\Omega}$ through Σ;

$$
\Delta t = \frac{4}{c^2}\int_\Sigma \Omega^k d\Sigma_k = \frac{4}{c^2}\int_\Sigma \mathbf{\Omega}\cdot d\mathbf{\Sigma}. \tag{10}
$$

Proof: *Eq.(10) follows from Eqs.(7) and (9) using Stokes theorem.*

Previous experimental verifications of this theorem have included measurements of the angular velocity of a spinning apparatus and the angular velocity of the earth. The experimental verifications required phase interference between the clockwise and counterclockwise paths for electromagnetic waves. The Sagnac measured interference phase shift $\Delta\Theta = \omega\Delta t$ for light of frequency ω is given by $\Delta\Theta = (4\omega/c^2)\int_\Sigma \mathbf{\Omega}\cdot d\mathbf{\Sigma}$. In the application of the Sagnac effect to deep space probes such as LISA, the time difference Δt *induced by a gravitational wave* is the central issue.

3 Gravitational Waves

A gravitational plane wave, which is not necessarily weak, may be theoretically described as follows[15]: (i) Start from flat space time with the metric

$$-c^2 d\tau_0^2 = -c^2 dt^2 + |d\mathbf{r}|^2. \tag{11}$$

(ii) Choose a unit spatial vector $\mathbf{n}$ for the propagation direction of the gravitational wave and define spatial coordinates parallel $r_\parallel$ and normal $\mathbf{r}_\perp$ to the propagation direction; i.e.

$$\mathbf{n}\cdot\mathbf{n} = 1, \quad r_\parallel = \mathbf{n}\cdot\mathbf{r} \quad \text{and} \quad \mathbf{r}_\perp = \mathbf{n}\times(\mathbf{r}\times\mathbf{n}). \tag{12}$$

(iii) The gravitational wave disturbance (at first described as a dimensionless "scalar" W) obeys the wave equation

$$\left\{\frac{1}{c^2}\left(\frac{\partial}{\partial t}\right)^2 - \Delta\right\}W(\mathbf{r},t) = \left\{\frac{1}{c^2}\left(\frac{\partial}{\partial t}\right)^2 - \left(\frac{\partial}{\partial r_\parallel}\right)^2 - \Delta_\perp\right\}W(\mathbf{r},t) = 0, \tag{13}$$

where $\Delta W \equiv (div\ \mathbf{grad})W$. (iv) The *plane wave* solution of Eq.(13) has the form

$$W(\mathbf{r},t) = \Psi\big(\mathbf{r}_\perp, t - (r_\parallel/c)\big) = \Psi\big(\mathbf{r}_\perp, t - (\mathbf{n}\cdot\mathbf{r}/c)\big), \tag{14}$$

with Ψ obeying Laplace's equation in the normal plane coordinates

$$\Delta_\perp\Psi(\mathbf{r}_\perp, t) = 0. \tag{15}$$

(v) Finally, the proper time in the presence of the gravitational wave is given by

$$\begin{aligned}
-c^2 d\tau^2 &= -c^2 d\tau_0^2 - W(dr_\parallel - cdt)^2, \\
-c^2 d\tau^2 &= -c^2 dt^2 + |d\mathbf{r}_\perp|^2 + dr_\parallel^2 - W(dr_\parallel - cdt)^2, \\
-c^2 d\tau^2 &= -(1+W)c^2 dt^2 + |d\mathbf{r}_\perp|^2 + (1-W)dr_\parallel^2 + 2Wcdtdr_\parallel.
\end{aligned} \tag{16}$$

The resulting proper time metric

$$-c^2 d\tau^2 = -(1+W)\left\{cdt - \left(\frac{W}{1+W}\right)dr_\parallel\right\}^2 + |d\mathbf{r}_\perp|^2 + \left\{\frac{dr_\parallel^2}{1+W}\right\}, \tag{17}$$

with W given in Eqs.(14) and (15), represents an *exact gravitational wave solution* to the vacuum Einstein curvature equation $\mathcal{R}_{\mu\nu} = \mathcal{R}^{\lambda}{}_{\mu\lambda\nu} = 0$. There is no *theoretical* requirement that the gravitational disturbance be small in the sense that $|W| \ll 1$. However it is expected that *experimental* gravitational wave disturbances will be very small indeed.

From Eqs.(5), (12) and (17) it follows that a gravitational wave carries a local flow velocity

$$\mathbf{u} = \frac{cW\mathbf{n}}{1+W} \ . \tag{18}$$

and local rotational velocity

$$\begin{aligned}
\mathbf{\Omega}(\mathbf{r},t) &= \frac{\mathbf{curl}\ \mathbf{u}(\mathbf{r},\mathbf{t})}{2} \ , \\
\mathbf{\Omega}(\mathbf{r},t) &= \frac{c}{2}\left\{\frac{\mathbf{grad}W(\mathbf{r},t)\times\mathbf{n}}{(1+W(\mathbf{r},t))^2}\right\}, \\
\mathbf{\Omega}(\mathbf{r},t) &= \frac{c}{2}\left\{\frac{\mathbf{grad}_{\perp}\Psi\big(\mathbf{r}_{\perp},t-(r_{\parallel}/c)\big)\times\mathbf{n}}{\big[1+\Psi\big(\mathbf{r}_{\perp},t-(r_{\parallel}/c)\big)\big]^2}\right\}.
\end{aligned} \tag{19}$$

With the spatial length scale

$$\begin{aligned}
d\ell^2 &= \gamma_{ij}dr^i dr^j = |d\mathbf{r}_{\perp}|^2 + \left\{\frac{dr_{\parallel}^2}{1+W}\right\} \ , \\
\gamma &= \det(\gamma_{ij}) = \frac{1}{1+W} \ ,
\end{aligned} \tag{20}$$

the cross product in Eqs.(19) associated with the spatial metric is given by

$$(\mathbf{a}\times\mathbf{b})^k = \left(\frac{a_i b_j}{\sqrt{\gamma}}\right)\epsilon^{ijk}. \tag{21}$$

The Sagnac effect time delay Δt induced by a gravitational wave is rigorously determined by Eqs.(10), (15) and (19). Let us examine the consequences of the above results for a plane polarized gravitational wave. The problem at hand is to find an appropriate solution to Eq.(15).

4 Gravitational Wave Rotational Velocity Flux

Let us assume that before the gravitational wave impinges on the laser interferometer space antenna. The center of area $\mathbf{R}$ of the triangle is given by

$$\mathbf{R}\ \Sigma = \int_{\Sigma}\mathbf{r}\ d\Sigma. \tag{22}$$

If e denotes the symmetric polarization tensor of the gravitational wave, then we have

$$\mathsf{e}\cdot\mathbf{n} = \mathbf{n}\cdot\mathsf{e} = 0 \ \text{ and } \ tr\{\mathsf{e}\} = 0. \tag{23}$$

The solution to Eq.(15) corresponding to gravitational wave reads

$$\Psi(\mathbf{r}_\perp, t) = \{\mathbf{r} - \mathbf{R}\}_\perp \cdot \mathsf{e} \cdot \{\mathbf{r} - \mathbf{R}\}_\perp \, \mathcal{A}(t), \tag{24}$$

where the non-vanishing curvature tensor elements are proportional to the amplitude $\mathcal{A}\big(t - (r_\parallel/c)\big)$. Eqs.(19) and (24) imply (for a weak gravitational wave)

$$\mathbf{\Omega} = c \, \mathsf{e} \cdot \{\mathbf{r} - \mathbf{R}\}_\perp \times \mathbf{n} \, \mathcal{A}\big(t - (r_\parallel/c)\big) + \dots \, . \tag{25}$$

Thus, to lowest order in the gravitational wave curvature and to lowest order in the gravitational wave frequency (c/λ) we find that

$$\begin{aligned} c\Delta t &= \frac{4}{c}\int_\Sigma \mathbf{\Omega} \cdot d\mathbf{\Sigma}, \\ c\Delta t &\approx 4\mathcal{A}(t) \int_\Sigma \mathsf{e} \cdot \{\mathbf{r} - \mathbf{R}\}_\perp \times \mathbf{n} \cdot d\mathbf{\Sigma}, \\ c\Delta t &\approx 4\mathcal{A}(t) \, \mathsf{e} \cdot \left(\int_\Sigma (\mathbf{r} - \mathbf{R}) d\Sigma \right) \times \mathbf{n} \cdot \mathbf{N}, \end{aligned} \tag{26}$$

wherein the unit vector $\mathbf{N}$ is normal to the area $d\mathbf{\Sigma} = \mathbf{N} d\Sigma$.

Theorem 2: (The Gravitational Wave Sagnac Effect)

The quasi-static ($\lambda^2 >> \Sigma$) *weak* ($|W| << 1$) *gravitational wave Sagnac effect is null*;

$$\Delta t \approx 0. \tag{27}$$

Proof: *Eq.(27) follows from Eqs.(22) and (26).*

The above theorem is the central result of this work.

5 Conclusions

Let us consider the physical meaning of the local drift velocity $\mathbf{u}$. If we examine Eq.(5), then the null proper time $d\tau = 0$ of a massless photon traveling on a light ray path yields a light ray optical path length

$$dL = cdt = \frac{d\ell}{\sqrt{h}} + \frac{\mathbf{u}}{c} \cdot d\mathbf{r}. \tag{28}$$

For a path $\mathcal{P}$ in a quasi-static spatial metric $d\ell^2 = \gamma_{ij} dr^i dr^j$ one may compute the optical path length

$$L[\mathcal{P}] = c \int_\mathcal{P} dt = \int_\mathcal{P} \left(\frac{d\ell}{\sqrt{h}} + \frac{\mathbf{u}}{c} \cdot d\mathbf{r} \right). \tag{29}$$

The Fermat principle of least time, $\delta \int_\mathcal{P} dt = 0$ yields the principle of least optical path length

$$\delta L[\mathcal{P}] = 0. \tag{30}$$

Eqs.(29) and (30) yield the light ray equation in a quasi-static gravitational field. That the Fermat variational principle Eq.(30) yields experimental light propagation is evident ever since the very first experiments on the gravitational bending of light rays. In Eq.(29), the index of refraction of the gravitational field

$$n = \frac{1}{\sqrt{h}} \ , \quad \text{yields} \quad L_0[\mathcal{P}] = \int_{\mathcal{P}} \frac{d\ell}{\sqrt{h}} = \int_{\mathcal{P}} n d\ell. \tag{31}$$

The additional optical path length ($L = L_0 + \Delta L$), as given by

$$\Delta L[\mathcal{P}] = c\Delta t[\mathcal{P}] = \int_{\mathcal{P}} \frac{\mathbf{u}}{c} \cdot d\mathbf{r}, \tag{32}$$

is due to the local drift velocity $\mathbf{u}$. The meaning of the local drift velocity becomes evident if we consider two photons with equal but opposite momenta

$$\mathbf{p}_+ = \hbar\mathbf{k} \quad \text{and} \quad \mathbf{p}_- = -\hbar\mathbf{k}. \tag{33}$$

The velocities of the two photons, $\mathbf{v}_+$ and $\mathbf{v}_-$, will not *quite* be equal and opposite. In the notation of Eq.(4), we have

$$(\mathbf{v}_+ + \mathbf{v}_-) = \left\{ \frac{2h\mathbf{u}}{1 - h|(\mathbf{u}/c)|^2} \right\} \approx 2\mathbf{u} \approx 2cW\mathbf{n} \tag{34}$$

to linear order in the gravitational wave amplitude W.

A photon moving parallel to the gravitational wave need not have the same speed as a photon moving anti-parallel to the gravitational wave since the drift velocity $\mathbf{u} = \{cW\mathbf{n}/(1+W)\}$ is non-vanishing is some regions of space. The photon velocity in the clockwise and counterclockwise orientations will have along the two paths (in general) $\mathbf{v}_+ \neq -\mathbf{v}_-$. However, for a plane gravitational wave incident upon the triangle with $W(\mathbf{R}, t) = 0$, we have varying directions for $(\mathbf{v}_+ + \mathbf{v}_-)$. The resulting flux $(c^2\Delta t/4) = \int_\Sigma \mathbf{\Omega} \cdot d\mathbf{\Sigma} \approx 0$. Experiments which measure gravitational waves via the Sagnac effect are thereby rendered unlikely to succeed.

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